

A novel scheme for entanglement engineering in a fermionic system

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Abstract

In this letter, we present a novel scheme to engineer the entanglement in a fermionic system, which is modeled by a minimally three-site Hubbard model. It is found that, in this type of system, we can have two free parameters. One is used to tune the entanglement and the other to switch on or off the above tuning function. The whole process is much like what happens in a traditional transistor switch, where the main current between the emitter and collector can be turned on or off by the base current.

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Quantum entanglement, is considered to be one of the key concepts in quantum mechanics, which has no classical analog. Ever since first noted by Einstein, Podolsky, and Rosen (EPR) [1] and Schrödinger [2], the entanglement has received wide experimental and theoretical attentions [3,4,5,6,7,8]. Recently, it is believed that quantum entanglement is a crucial resource in quantum information processing (QIP) [9], such as quantum computations, quantum state transfer [10], dense coding [11,12], quantum communication [13] and quantum cryptography [14]. Hence the present research upon entanglement has moved from philosophical debates to applied fields and concrete theoretical study. And generally speaking, the entanglement "engineering" has become the core of QIP, no matter whether it is realized with optical systems or with atomic ones.

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In condensed matter physics, correspondingly, much of the theoretical investigations are focused upon spin or itinerant fermionic systems, such as the transverse Ising model [15], Heisenberg model [16] and Hubbard model [17]. A marvelous progress in this direction is the discovery of the close relationship between the entanglement and the quantum phase transition (QPT) [15,18,19]. It has been found that there is an abrupt change of the entanglement over the quantum critical point. The significance of this result lies not only in its physical implication to describe the long-range correlations in the many-body systems, but also in its potential application to control the entanglement in QIP. For example, we can use this effect to realize an entanglement switch by moving the system from one side of the critical point to the other side. During this process, the entanglement will jump between two stable values, just like a switch. This nice idea has been put forward for some time now [15,19,20,21]. But in its practical application, we can observe two obstacles. Firstly, QPT only happens in thermodynamic limit, which means that a large number of particles will be involved in the process. This will bring much trouble in pinning down the entanglement among finite number of particles or sites. Secondly, the infinite size of the system will put a severe limit upon the response time of the entanglement switch, which will influence the whole QIP efficiency. These two problems are inherent in QPT and hard to be solved so long as QPT is involved. But the idea of entanglement switch has motivated us to put forward a new scheme, which could not only realize the same purpose as QPT has been intended to achieve, but also give us more. In the following, a tight-binding Hubbard model will be used as a generic example to illustrate our idea.

For QPT, the principal trouble for efficient engineering of the entanglement lies in its infinite number of freedoms. Hence, in our model, we will only consider finite number of sites. Moreover if only the engineering of the well-understood two-particle entanglement is concerned, we need to consider at least a two-site Hubbard model. In the past few years, torrents of experimental and theoretical work have been carried out for two-site systems under various conditions [22,23,24,29]. But for our purpose, as we shall see, two sites are not enough to realize the switch function. Hence, an extra site will be included, which plays a crucial role in controlling the entanglement engineering between the other two sites. Actually, this kind of three-site system can demonstrate fruitful static and dynamic effect as shown in Ref. [25] for a spin chain. Here, for the 3-site Hubbard model, we will not investigate all its quantum behavior, but focus upon the effect of entanglement transistor switch as we name it. Before going to the details, the basic engineering process will be summarized as follows. Namely, two free parameters will be chosen first. Then one is used to switch or tune the entanglement and the other to turn off or on the above function. We call these two parameters as "tuning parameter" and "control parameter", respectively. Compared with QPT, this is a few-body system, in which the two-site entanglement could be well defined and analyzed for QIP.

Most importantly, the response time will be much reduced.

Fig. 1 schematically shows the system setup in two different geometries. For the cyclic geometry in Fig. 1(a), the Hamiltonian is,

$$H = -t \sum_{\sigma} \left(c_{1\sigma}^{\dagger} c_{2\sigma} + c_{2\sigma}^{\dagger} c_{3\sigma} + c_{3\sigma}^{\dagger} c_{1\sigma} + h.c. \right) + U_0 (n_{1\uparrow} n_{1\downarrow} + n_{2\uparrow} n_{2\downarrow}) + U n_{3\uparrow} n_{3\downarrow}, \quad (1)$$

where t stands for the hopping between the nearest neighboring sites, $\sigma = \uparrow, \downarrow$ is the electron spin, $c_{i\sigma}^{\dagger}$ and $c_{j\sigma}$ are the creation and annihilation operators on the i th site and U and U_0 are the on-site Coulomb interactions. For the linear chain geometry in Fig. 1(b), the Hamiltonian is a bit different,

$$H = -t \sum_{\sigma} \left(c_{1\sigma}^{\dagger} c_{2\sigma} + c_{2\sigma}^{\dagger} c_{3\sigma} + h.c. \right) + U_0 (n_{1\uparrow} n_{1\downarrow} + n_{2\uparrow} n_{2\downarrow}) + U n_{3\uparrow} n_{3\downarrow}, \quad (2)$$

In fact, the two geometries are equivalent to two different boundary conditions. Fig. 1(a) is for a periodic boundary condition and Fig. 1(b) for a open boundary one. The entanglement between the first two sites inside the dotted box is what we intend to engineer. And the parameters related to the first two sites are assumed to be the same. Due to the itinerant feature of the system, each site has four possible states, which could be taken as a generalized version of two-state qubit. The four states can be expressed as $|0\rangle, |\uparrow\rangle, |\downarrow\rangle, |\uparrow\downarrow\rangle$. Depending upon the number and the spins of the electrons contained in the system, some of the four states might be depressed. By convention, for this kind of itinerant fermionic system, the von Neuman entropy will be used to measure the entanglement, i.e. if a two-part system is described by a density matrix ρ , the entanglement between the two parts can be measured by the following expression,

$$E = -Tr(\rho_1 \log_2 \rho_1), \quad (3)$$

where $\rho_1 = Tr_2(\rho)$ denotes the tracing over the freedoms from the second part of the system.

From Eq. (1) and Eq. (2), it is also apparent that, if we scale all the energies by t , only two free parameters are left, i.e. U/t and U_0/t . To see the role played by site 3, which provides us the control function through the parameter U/t , we first cut it off from the system. In this situation, the density matrix ρ can be written as $|\psi_0\rangle \langle\psi_0|$ with ψ_0 to be the ground state wave function. In order to have a system as simple as possible, we also need to fix the particle number N and the total electron spin S_z , both of which are good quantum numbers. The cases with $N = 0$, $N = 1$, $N = 3$ and $N = 4$ are trivial since the corresponding ground state is independent of U_0/t , which is also true for $N = 2$ with $S_z = \pm 1$. Hence the only nontrivial case is for $N = 2$ with $S_z = 0$, i.e., one electron spin up and the other down. For this well-defined case, the numerical method

of exact diagonalization is used to calculate the entanglement. The results are presented in Fig.2(a). It can be easily seen that $U_0/t = 0$ corresponds to the maximal entanglement with $E = 2$, which is a direct consequence of equal superposition of the configuration $|(\uparrow\downarrow)_1(0)_2\rangle, |(0)_1(\uparrow\downarrow)_2\rangle, |(\uparrow)_1(\downarrow)_2\rangle$ and $|(\downarrow)_1(\uparrow)_2\rangle$ in the ground state. Here $(\)_i$ denotes the basis state of the i th site. The corresponding single-site density matrix is

$$\rho_1 = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4)$$

As $U_0/t \rightarrow \infty$, the double occupancy is forbidden due to the strong on-site repulsion and the ground state is a superposition of $|(\uparrow)_1(\downarrow)_2\rangle$ and $|(\downarrow)_1(\uparrow)_2\rangle$ with equal coefficient, which leads to $E = 1$. Similarly, as $U_0/t \rightarrow -\infty$, the single occupancy is depressed and only $|(0)_1(\uparrow\downarrow)_2\rangle$ and $|(\uparrow\downarrow)_1(0)_2\rangle$ are equally superposed, which also leads to $E = 1$. All these results are consistent with those in Ref.([17]). It should be noted that, although in this case the entanglement can be continuously tuned from minimum 1 to maximum 2, no switch function as found in QPT can be realized since the entanglement demonstrates no distinct saturation value.

The whole scenario changes when a third site is included. The electron number N and the total spin S_z keeps the same as above, namely, we still have two electrons with one electron spin up and one spin down. Now the left two-site subsystem will be actually in a mixed state if we reduce site 3 from the density matrix $\rho = Tr_3 |\phi_0\rangle \langle \phi_0|$, where $|\phi_0\rangle$ is the ground state of the whole 3-site system. Fig. 2(b) shows a typical result for positive control parameter U/t . A great difference from Fig. 2(a) is that the entanglement now displays two distinct saturation values for positive U/t , just like what happens in QPT. This is exactly what we are looking for, from which an entanglement switch can be realized. And within a narrow region around $U_0/t = 0$, E can be continuously tuned. The saturation values for the cyclic case can be explained as follows. First, when U_0/t is positive enough, satisfying $U_0/t > U/t$, all the double occupancy on the three sites are disfavored in energy. Then the two electrons will sit on different sites with all possibilities. Due to the spatial symmetry of the system, each site has equal probabilities to be in the states $|\uparrow\rangle, |\downarrow\rangle, |0\rangle$. Hence, the corresponding entanglement is $E = \log_2 3 = 1.58$. By similar reasoning, we can have $E = 1$ when $U_0/t \ll U/t$ due to the equal probability for each site to be in the states of $|\uparrow\downarrow\rangle$ and $|0\rangle$. In the linear chain case, the same feature exists except that the saturation values are a bit different, which results from the asymmetric properties of the three sites.

Besides the above discussed switch function, another interesting static charac-

teristic is shown in Fig. 2(c). Namely, as U/t is negative, E keeps to be zero in a wide region of U_0/t , the width of which depends upon the control parameter U/t . Generally, $U_0/t > U/t$ is required to make this phenomenon appear. This is a consequence of the competition among the on-site interactions at different sites. Namely, if the on-site attractions at site 3 overwhelms the attraction at site 1 and 2, the two electrons will tend to be localized on site 3. So for the left two sites, there is only one possible state, i.e. $|(0)_1(0)_2\rangle$, which naturally leads to zero entanglement.

Through the above analysis, it can be seen that the role played by site 3 is crucial. Depending upon the sign and magnitude of the control parameter U/t , it can either act as a barrier forbidden the double occupancy or acts as an attractor freezing both the electrons on itself, thus providing different boundary conditions for the left two sites. By utilizing this kind of constraints, a novel scheme to engineer the entanglement between the left two sites can be formed. First, when U/t is positive enough, for example $U/t \sim 80$, the entanglement can be tuned between two different saturation values by varying U_0/t . Then by letting U/t negative enough, for example $U/t \sim -80$, the system will be driven into a cut-off status with zero entanglement. Because the whole picture is much like what happens in a traditional transistor switch in electronics, we called it "transistor switch" effect of the entanglement. Schematically, the on-site interaction U/t on site 3 can be compared to the base current and the entanglement to the main current between the emitter and the collector. Hence, the tuning function of the entanglement through U_0 can be switched on or off by setting U/t at different values, just as the flow of the main current can be switched on or off by properly setting the base current. This control mechanism can find potential applications in QIP. For example, it can be used as a memory or storage media of quantum information characterized by the entanglement. One of the advantages of this scheme is that the different properties of the entanglement make the whole mechanism very robust against the fluctuations of the tuning parameter U_0/t . Moreover, the stored quantum information can be easily erased just by letting U/t at a value more negative than U_0/t .

So far, we have only investigated a limited region of the parameter space of U_0/t and U/t for our purpose. The entanglement variation in the whole parameter space is plotted in Fig. 3(a) for cyclic case. The results for the linear-chain case are similar and not shown. From the figure, we can easily see how the transistor switch effect comes out when the parameters are independently varied along a specific route on the $U/t - U_0/t$ plane. Actually, if we can change U_0/t and U/t together, different forms of the entanglement variations can be realized, which will lead to various interesting engineering mechanism of the entanglement.

As we know, for any quantum device, the decoherence problem is a must to be

discussed. Since we are using a toy model to illustrate the main idea, we will only discuss the temperature-induced decoherence here. Once the thermal effect is considered, the measure of the entanglement becomes,

$$E = Tr_{2,3} \left[\frac{\sum_i e^{-\frac{\epsilon_i}{kT}} |\varphi_i\rangle \langle \varphi_i|}{\sum_i e^{-\frac{\epsilon_i}{kT}}} \right], \quad (5)$$

where T is the temperature, k is the Boltzmann constant, $|\varphi_i\rangle$ denotes the eigenfunction of the whole system with ϵ_i to be the corresponding eigenvalues and $Tr_{2,3}$ means the reduction of the freedoms related to site 2 and site 3. It is apparent that as $T \rightarrow 0$, only the ground state contributes to the sum and this definition goes to Eq. (3), as expected. The results are given in Fig. 3(b) and 3(c) for two different temperatures. It can be seen that the entanglement does not change much when the temperature is low enough, i.e. $kT/t \ll U/t$ or U_0/t . This is quite understandable since we are talking about a ground state property and the finite energy gap to the first excited state will help to guarantee some robustness of the entanglement over the thermal fluctuations. As we increase the temperature from zero, the biggest modification always starts from the border between the saturation plateaus and then extends to the other regions. This is due to the fact that the saturation border normally runs along $U/t = 0$ or $U_0/t = 0$, which sets the energy upper threshold for the ground state stability to a relatively low scale. And as $T \rightarrow \infty$, all the possibilities of the states will be excited by the thermal fluctuations. Hence the entanglement goes to a flat maximum, making all the engineering impossible.

As mentioned before, what we discussed here is a toy model. But due to the rapid technology progress related to quantum dots [26] and especially cold atoms in optical lattices [27], people have been able to readily manipulate single cold atoms in a well-controlled optical lattice or single ion and electron in different kinds of wells. For example, in Ref.([28]), the system with cold atoms loaded into an optical lattice is used to mimic the Hubbard model. The experimentally controllable parameters, such as the atom scattering length and optical potential depth, are directly mapped to the Hubbard hopping and on-site interaction terms. From the mapping diagram in Ref.([28]), it can be easily seen that most of the parameter space investigated in this paper can be covered by the present technology. Hence we are expecting a prompt experimental realization of the transistor switch effect.

In summary, in this letter, we have put forward a novel scheme to engineer the entanglement by using a three-site Hubbard model. The whole idea is much like the traditional transistor switch, i.e. the entanglement switching or tuning function can be turned on or off by an external well-controllable parameter. Hence, we have not only realized what QPT has been originally intended to do, but also gone a step further by introducing an extra parameter to control

its functioning. The influence of the thermal fluctuations upon the scheme has also been investigated. This scheme should have potential applications in the QIP, especially for making entanglement control devices, such as quantum memory or quantum storage media.

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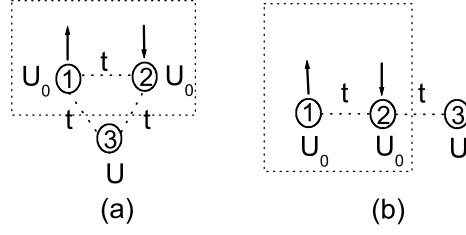


Fig. 1. Schematic diagram of the 3-site Hubbard model having two electrons with different spins. (a) is for a cyclic case corresponding to periodic boundary conditions and (b) for a linear chain case corresponding to open boundary conditions. The entanglement between the two sites in the dotted box are engineered by varying U/t and U_0/t . t is taken to be a scaling parameter of the energy in our work.

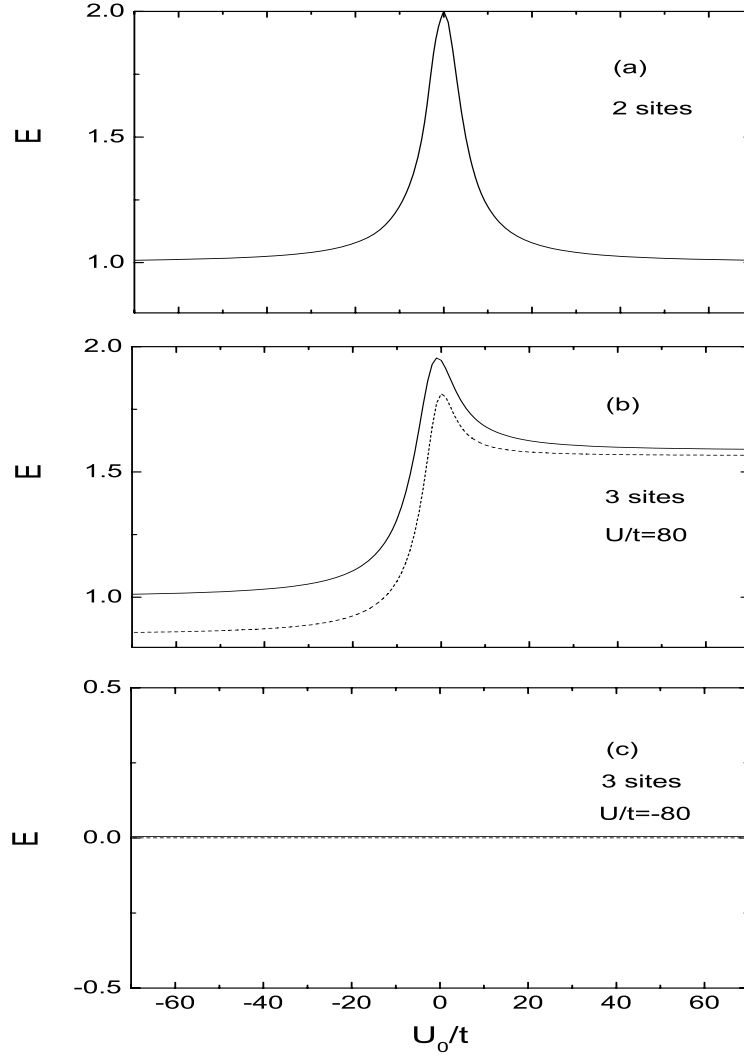


Fig. 2. Entanglement variations with respect to the tuning parameter U_0/t . (a) is for the two-site case. (b) and (c) are for the three-site case with positive and negative control parameter U/t . In both (b) and (c), the solid line represents the cyclic case and the dotted line the linear chain case.

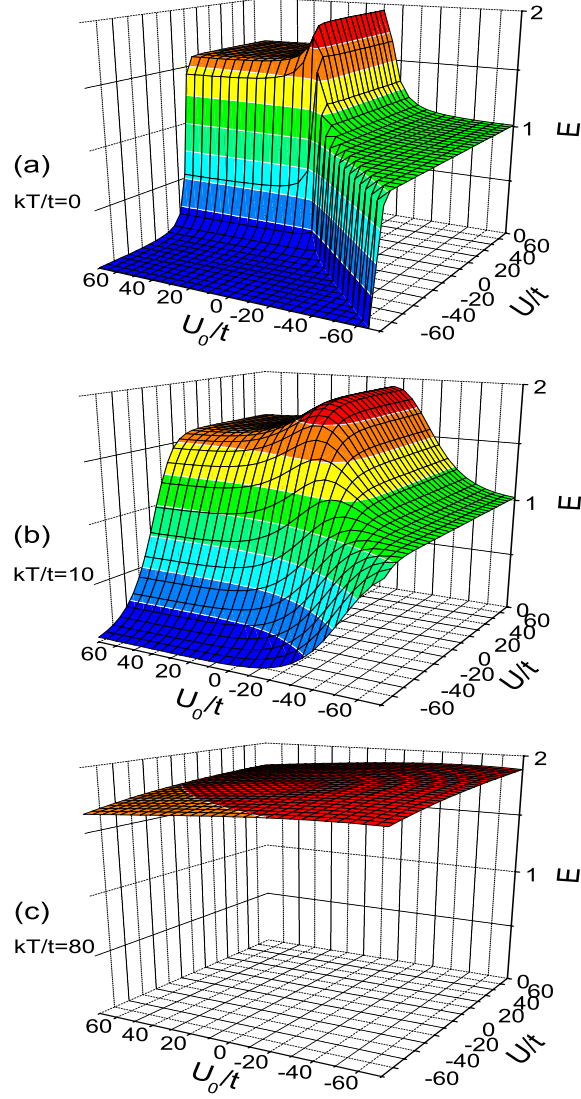


Fig. 3. Entanglement variations against U/t and U_0/t for the 3-site Hubbard model under different temperatures kT . (a) $kT/t = 0$. (b) $kT/t = 10$. (c) $kT/t = 80$.